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EDITED BY

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WITH THE CO-OPERATION OF

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A PROGRESSIVE INCOME TAX.

BY PROFESSOR H. S. CARSLAW, Sc.D.

IN the *Gazette* of May, 1916, I described the scheme of taxation embodied in the "Income Tax Act, 1915" (Commonwealth of Australia). Some mild censure of the extremely complicated schedules with their curves was expressed. The confusion between the rate of tax at the xth pound and the average rate on £x was pointed out. If the tax on an income of £x is T pence, the rate of tax at the xth pound is $\frac{dT}{dx}$, and the average rate on £x is $\frac{T}{x}$, in pence per pound.

At the end of 1916 a new Act (Income Tax Act, 1916) was passed with practically no discussion. The criticism on the complexity of the scheme had no effect on the officials concerned. But the remarks on the confusion between the terms "rate of tax" and "average rate" have borne curious fruit.

In Schedule I., dealing with income from personal exertion, the first clause now reads:

> For so much of the whole taxable income as does not exceed £7600 the rate of tax per pound sterling shall be threepence and three eight-hundredths of one penny where the taxable income is One pound sterling, and shall increase uniformly with each increase of One pound sterling of the taxable income by three eight-hundredths of one penny.

But the final clause, giving the formula which is supposed to correspond to this rate, is as follows:

> The average rate of tax per pound sterling for so much of the taxable income as does not exceed £7600 may be calculated from the following formula:
>
> R = average rate of tax in pence per pound sterling.
>
> I = taxable income in pounds sterling.
>
> $R=\left(3+\frac{3}{800}I\right)$ pence.

It has been pointed out to the authorities that the difference between the amount which the first clause says we must pay, and that which the last clause says we may pay, is not a slight one. On an income of £800 it is a matter of £5; on £4000, of £125; and on £7600, the limit in this section, of £451 5s.

There is little doubt that the Judges of the High Court will now, it may be for the first time, learn something about the Differential Calculus!

Schedule II. deals, as before, with income derived from property. The first section is improved by the insertion of the word "average" before "rate," and now reads:

For income of a taxable value not exceeding £546 the average rate of tax per pound sterling shall be calculated from the following formula :

R = average rate of tax in pence per pound sterling.

I = taxable income in pounds sterling.

$$R=\left(3+\frac{I}{181\cdot 058}\right) \text{ pence.}$$

In the second section we come again to the curve of the second degree. This section now reads as follows :

For income of a taxable value exceeding £546 but not exceeding £2000 the rate of tax shall be calculated in the following manner :

The rate of tax on each additional pound shall increase continuously with the increase of the taxable value of the income in a curve of the second degree in such a manner that the increment of tax for one pound increase of taxable income shall be :

at a taxable income of £546	11·713 pence
at a taxable income of £600	12·768 pence
at a taxable income of £700	14·672 pence
at a taxable income of £800	16·512 pence
at a taxable income of £900	18·288 pence
at a taxable income of £1000	20·000 pence
at a taxable income of £1500	27·600 pence
at a taxable income of £2000	33·600 pence

Comparison with the 1915 Act (cf. vol. viii. p. 257) will show that a number of changes have been made, but they cannot be called improvements. I do not know what is meant by the term "rate of tax on each additional pound." But I do know that in the Tables issued by the Commissioners the tax is found by integrating the rate, and that if the rate is r pence per pound at £x, r is given by the equation

$$r=\frac{23\cdot 2}{10^3}x-\frac{3\cdot 2}{10^6}x^2. \quad \text{(Cf. \textit{Math. Gaz.} vol. viii. p. 257.)}$$

It can be verified that the figures in the section are, in fact, the rates at the values of x named. They are not the increments.

Similar remarks might be made about the third section.

In the 1916 Act, Schedule III. deals with composite incomes, derived partly from personal exertion and partly from property. Without entering into details, I may say that the elaborate device of the gradually increasing rate is given up. A fair method of determining the tax would be to take the income from personal exertion first, and assess it according to the rate for such income ; then, in dealing with the income from property, to start with the rate of the second schedule corresponding to the income already reached, and proceed along the rate-curves of that schedule from that point. With these curves this method is out of the question. A rough and ready method is chosen, destroying the whole claim of the Act to a mathematical basis.

There is much to be said for the idea of a continuously ascending rate of tax, but for practical purposes it should be kept straight ($r=a+bx$). If the first straight line is not steep enough, when the incomes get fairly large a steeper one can be chosen, and later a steeper one still. And the same remark applies to both kinds of income. Also the difficulty of fairly dealing with composite incomes is easily met.

However, the idea of a continuously ascending rate of tax is certainly beyond the average taxpayer. He need not be bothered with it, and the schedules could contain merely a statement of the amount to be paid on incomes falling within the different sections.

The schedules of the Australian Acts, from which I have been quoting, might as well have been written in Chinese, so far as our legislators, who passed the Acts, were concerned. They would have understood them equally well. And the same is true of 99 per cent. of the taxpayers. This fact alone is sufficient to prove their unsuitability. H. S. Carslaw.

April 17, 1917.

MATHEMATICAL NOTES.

517. [C. 2. a. j.] **496.** ***Integral Calculus. Squaring the Circle and Hyperbola.***

The utility of Mr. Langley's method of the integration of sec ϕ was shown in its geometrical and dynamical application in the *Mathematical Gazette*, Dec. 1916, March 1917; and it may be extended to the integration of the other functions in showing the geometrical interpretation.

1. Mr. Langley has given the integral of sec ϕ, and to complete the series on Fig. 1, draw AQS, Aqs to meet OB in S, s. Then Aq cuts the circle at q and OB in s, or MQ in m, at the same angle $\frac{1}{2}\phi$; so that $Qm=Qq$:

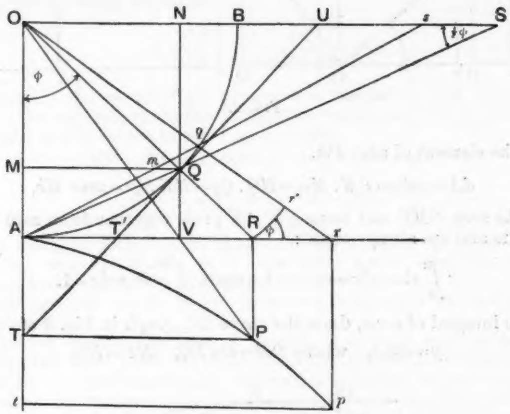

FIG. 1.

$$(1)\qquad \frac{d\phi}{\sin\phi}=\frac{Qq}{QM}=\frac{Qm}{QM}=\frac{Ss}{SO};$$

and by the former definition of the Napierian logarithm to base e,

$$(2)\qquad \int^{\frac{1}{2}\pi} \operatorname{cosec}\phi\, d\phi=\log\frac{OS}{OB}=\log\cot\tfrac{1}{2}\phi$$

$$=\log\frac{OU+UQ}{OB}=\log(\operatorname{cosec}\phi+\cot\phi).$$

Draw the arc Rr', centre O, meeting Or in r'; then

$$(3)\qquad d\phi=\frac{Rr'}{OR},\quad OT=OR,\quad Tt=r'r=Rr'\tan\phi,$$

ultimately in the small triangle Rrr'; and

$$(4)\qquad \tan\phi\, d\phi=\tan\phi\,\frac{Rr'}{OR}=\frac{r'r}{OR}=\frac{Tt}{OT},$$

$$(5)\qquad \int_0 \tan\phi\, d\phi=\log\frac{OT}{OA}=\log\sec\phi.$$

Similarly for the integral of cot ϕ.

2. For the integral of $\sin x$, draw the curve OPB in Fig. 2, graph of the equation

(1) $$y = \sin x, \quad \text{where } OM = \text{arc } AQ, \quad MP = HQ.$$

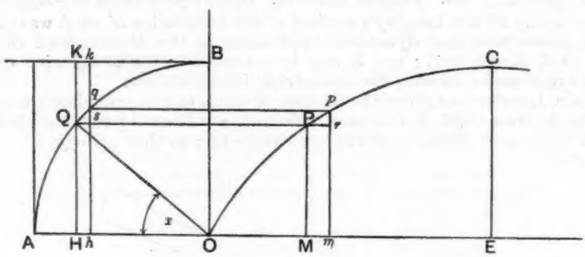

Fig. 2.

Then the element of area Pm,

(2) $$dA = y\,dx = PM \,.\, Mm = HQ \,.\, Qq = HK \,.\, Qs = \text{area } Hk,$$

so that the area OMP and rectangle AK grow together from zero by equal increments and are always equal:

(3) $$\int_0^x \sin x\,dx = \text{vrs}\,x = 1 - \cos x, \quad \int_0^{\frac{1}{2}\pi} \sin x\,dx = 1.$$

For the integral of $\cos x$, draw the curve BC, graph in Fig. 3 of

(4) $$y = \cos x, \quad \text{where } OM = \text{arc } BQ, \quad MP = HQ.$$

Fig. 3.*

The element of area Pm,

(5) $$dA = y\,dx = PM \,.\, Mm = HQ \,.\, Qq = HK \,.\, Qs = \text{area } Hk;$$

and the area $OMPB$ and rectangle OK grow together from zero by equal increments and are always equal:

(6) $$\int_{0,\,0}^{x,\,\frac{1}{2}\pi} \cos x\,dx = \text{siu } x, \ 1.$$

* In fig. 3, M is on the circle, which is exceptional.

In the cycloid *RBS*, generated by the rolling of the circle on the line through B' parallel to Ox:

(7) $$QR = \text{arc}\, BQ = x, \quad Q'R' = \text{arc}\, BQ' = \pi - x, \quad QR + Q'R' = \pi.$$

As x increases by dx, $QR+Q'R'$ sweeps out area $\pi . Nn$; so that the area between the semi-circle BAB' and the semi-cycloid $BRR'D'$ is π, twice the area of the semi-circle.

The area of the whole cycloid is then three times the area of the circle on BB'.

The sinusoidal curve *BCD* was called formerly the *companion of the cycloid.*

In a quarter wave of the sinusoid, $y = \sin x$ or $\cos x$, as BC in Fig. 3, with $OC = \frac{1}{2}\pi . OB$, the mean ordinate

(8) $$y = \frac{OB}{\frac{1}{2}\pi} = \frac{OB^2}{\text{quadrant arc } AB}.$$

Mr. Langley has resumed his geometrical method in 511, *Math. Gazette*, p. 90, in the equivalent of the integration requiring the inverse sine or cosine; and this is shown in our Fig. 2, where

$$MP = HQ = y, \quad OB = a, \quad OH = \sqrt{(a^2 - y^2)}:$$

(9) $$\frac{dy}{\sqrt{(a^2-y^2)}} = \frac{qs}{OH} = \frac{Qq}{OQ} = dx = d \sin^{-1}\frac{y}{a}.$$

3. The Integral Calculus to many minds is easier to grasp than the Differential, and historically it is over 2000 years older, the ideas and operations being familiar to Archimedes.

And Integration can be made visible to the eye in a way not possible of Differentiation.

We cannot observe the rate of growth of a body such as a tree, but returning after some years the total integrated growth is noticeable.

A treatise on the Integral and Differential Calculus, I.C. and D.C., would be too great an innovation to be accepted for instruction, giving Integral the first place, except from the hand of such an innovator as Professor Perry.

But the simplest algebraical integral, of x^n as $\frac{x^{n+1}}{x+1}$, is a hard saying, not capable of a pictorial representation except through the Differential Calculus; and it must be approached through an upper and lower limit.

Stated in the language of the average or mean value, we translate the theorem that

(1) $$\int_0 y\, dx = \frac{ax^{n+1}}{n+1} = \frac{xy}{n+1}, \quad \text{if} \quad y = ax^n,$$

as meaning that $\bar{y}$, the mean or average value of y, starting from $x=0$ in the corrected integral, is given by

(2) $$\bar{y} = \frac{y}{n+1}.$$

Thus in an Indicator Diagram, the area is read off, either by a Planimeter or by a rule such as Simpson's, of approximate quadrature; and then the mean ordinate, height, or breadth is to be deduced representing P, mean effective pressure (M.E.P.) in lb/inch^2; and P is used in calculating the H.P. for given stroke, L feet, piston area, A inches^2 and revolutions, N per minute, by the rule

(3) $$\text{H.P.} = \frac{PLAN}{33000}.$$

Between other limits, x_1 and x_2 of x, the integral (1) is

(4) $$\int_{x_1}^{x_2} y\,dx = \frac{x_2y_2 - x_1y_1}{n+1}, \quad \text{if} \quad y = ax^n,$$

(5) $$\bar{y} = \frac{1}{n+1}\,\frac{x_2y_2 - x_1y_1}{x_2 - x_1}.$$

Then the theorem is required of the Differential Calculus,

(6) $$y = ax^n, \quad \frac{dy}{dx} = nax^{n-1} = n\frac{y}{x}, \quad \frac{dy}{y} = n\frac{dx}{x},$$

meaning in practical language that 1 % increase in x gives n % increase to y. No need to dwell here on the variety of proofs.

Represented on a diagram in Fig. 4, with x_1, x_2 close together on each side of x, the rectangle $N_1Q_1 = Q_1R_2$, so that

(7) $$x_2y_2 - x_1y_1 = \text{gnomon } N_1P_2M_1 = \text{rectangle } N_1Q_1 + Q_1M_2$$
$$= \text{rect. } M_1R_2 = MR \,.\, M_1M_2.$$

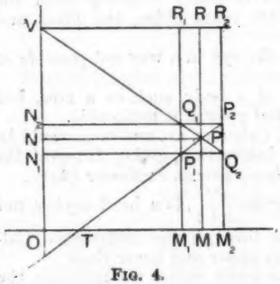

FIG. 4.

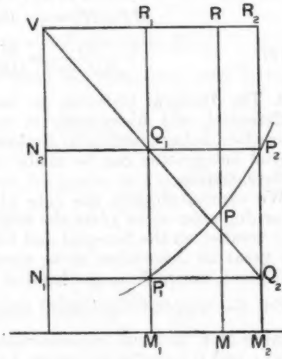

FIG. 5.

But by the theorem of the D.C.,

(8) $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{P_1Q_1}{P_1Q_2} = \tan NPV = \frac{NV}{NP} = \frac{PR}{OM},$$

(9) $$\frac{y_2 - y_1}{x_2 - x_1} \to \frac{dy}{dx} = n\frac{y}{x} = n\frac{MP}{OM},$$

(10) $$PR = n\,.\,MP, \quad MR = (n+1)MP,$$

(11) $$x_2y_2 - x_1y_1 = (n+1)MP\,.\,M_1M_2 = (n+1)y\,dx = (n+1)dA,$$

where dA is the element of area $M_1P_1P_2M_2$, as P may be taken as the mid-point of P_1P_2.

Then as P_1 and P_2 separate from P along the curve $y = ax^n$, as in Fig. 5,

(12) $$A = \int y\,dx = \frac{x_2y_2 - x_1y_1}{n+1},$$

(13) the area $M_1P_1P_2M_2 = \frac{MR.M_1M_2}{n+1} = MP.M_1M_2$,

where $MP = \bar{y}$ is the mean ordinate of the curve P_1PP_2,

(14) $$\bar{y} = \frac{1}{n+1}\,\frac{x_2y_2 - x_1y_1}{x_2 - x_1}.$$

4. In the usual physical applications, as of Thermodynamics, connecting the pressure y at volume x of a given quantity of gas, in the form

(1) $$y = \frac{a}{x^n},$$

the sign of n is changed, and the shape of the curve is as in Figs. 6 and 7 with V below N, and

(2) $$NV = \frac{-dy}{\frac{dx}{x}} = -x\frac{dy}{dx} = ny = n.ON;$$

and NV thus measures the cubical elasticity, ratio of $\left\{\begin{matrix}\text{increment}\\ \text{decrement}\end{matrix}\right\}$ of pressure, $\pm dy$, to the cubical $\left\{\begin{matrix}\text{compression}\\ \text{expansion}\end{matrix}\right\}$, $\mp\frac{dx}{x}$.

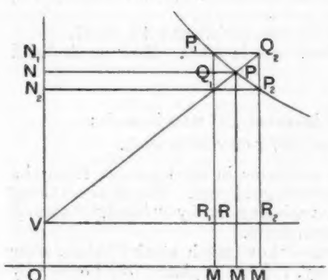

FIG. 6.

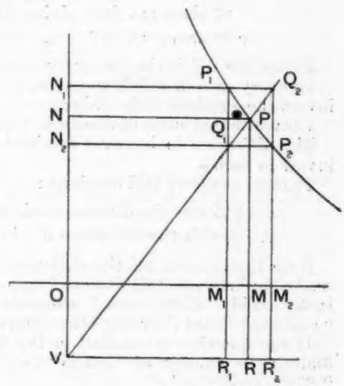

FIG. 7.

In Fig. 6, $n < 1$; and with P_1, P_2 close to P,

(3) $$PR = n.PM, \quad RM = (1-n)PM,$$

(4) $$x_2y_2 - x_1y_1 = RM.M_1M_2 = (1-n)PM.M_1M_2 = (1-n)dA,$$

and drawn apart again along the curve,

(5) $$\text{area } P_1M_1M_2P_2 = A = \int y\,dx = \frac{x_2y_2 - x_1y_1}{1-n}.$$

In Fig. 7, $n > 1$, and V is below O,

(6) $$x_1y_1 - x_2y_2 = RM.M_1M_2 = (n-1)PM.M_1M_2 = (n-1)y\,dx,$$

when P_1, P_2 are close to P; draw apart

(7) $$\text{area } P_1M_1M_2P_2 = A = \int y\,dx = \frac{x_1y_1 - x_2y_2}{n-1}.$$

In the separating case between Figs. 6 and 7, $n=1$, and the curve is the hyperbola. The Exponential Theorem is required then in accordance with the preceding definition employed in § 1 of the logarithm ; here

$$x_1y_1=x_2y_2=xy=c^2, \text{ a constant,} \tag{8}$$

$$A=\int_{x_1}^{x} y\,dx=\int\frac{c^2}{x}\,dx=c^2\log\frac{x_2}{x_1}. \tag{9}$$

In the Gas Equation of Thermodynamics, the temperature would be constant in the expansion along the hyperbolic curve, $n=1$.

In Fig. 6, with $n<1$, the temperature would rise as the gas expands, as in the *endothermic* state of a charge of powder during combustion.

June 6, 1917. G. GREENHILL.

518. [V. 1. a. δ.] *A Point in teaching Stocks.*

In determining which of two stocks is the better investment, given the market prices and the percentages paid, always take the *Product of the Prices.*

Which is the better investment, the 3 % at 69 or the 4 % at 87 ?

Invest in each £69 × 87.

This brings *either*

87 shares at £69, giving £3 × 87, *i.e.* £261 per annum,
or 69 shares at £87 „ £4 × 69, *i.e.* £276 „

Hence the 4 % is the better investment.

In every case in which an option as to the amount to be invested is given, invest the *Product of the Prices.*

I invest equal sums of money in the 3 % at 69, and in the 4 % at 87.

The difference in increase obtained from each is £300. How much did I invest in each ?

Proceed as above and continue :

£15 was the difference when I invested £87 × 69 in each ;
∴ £300 is the difference if I invest £87 × 69 × 20 in each.

Note that almost all the difficulty in problems in stocks arises from the confusion between "£100 stock" and so much gold coin. The obvious thing is to avoid "£100 stock" altogether, replacing it by "bond," "share," "voucher," "cent.," or any other convenient term.

If the question is couched in the form—"how much stock ?" then, after finding the number of "bonds" to be, say, $75\frac{1}{4}$, we replace "$75\frac{1}{4}$ bonds" by "£7525 stock." L. M.

W. E. HARTLEY, M.A.

H.M.S. VANGUARD

JULY 9, 1917

MR. HARTLEY co-operated in editing our "Problems and Solutions," Dec. 1900—May 1906

REVIEWS.

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies. By E. T. Whittaker. Second Edition. Cambridge: University Press, 1917. Pp. xii, 432. Price 15s. net.

This new edition of Prof. Whittaker's valuable and well-known treatise is not very greatly different from its first edition of 1904. The author has "endeavoured to give references to, and in some cases accounts of, the numerous original researches in Dynamics which have been published by various investigators since the first edition appeared." Further, he has "added some historical matter, and rewritten many sections" (p. v). The chief alterations will be mentioned further on. The sixteen chapters deal respectively with kinematical preliminaries; the equations of motion; principles available for the integration; the soluble problems of particle dynamics; the dynamical specification of bodies; the soluble problems of rigid dynamics; theory of vibrations; non-holonomic systems and dissipative systems; the principles of least action and least curvature; Hamiltonian systems and their integral-invariants; the transformation-theory of dynamics; properties of the integrals of dynamical systems; the reduction of the problem of three bodies; the theorems of Bruns and Poincaré; the general theory of orbits; and integration by trigonometric series. The chief alterations are a new explanation of the transformation-theory of dynamics and Hamilton's work—his earliest original work—on the characteristic function (pp. 288–92); the changes in the treatment of the motion of a body about a fixed point under no forces (pp. 144–52), which arise from Prof. Whittaker's opinion that the Jacobian elliptic functions are preferable to the Weierstrassian ones in numerical computations; and the short account (pp. 411–2) of Sundman's theorem on the regularisation of the problem of three bodies, which is meant to inspire the student to read Sundman's original paper.

In the following criticisms some of the points criticised belong to the first edition as well as his second one, but it seems relevant to point out statements which, it seems to me, ought to have been altered. The addition to this edition (p. 26) stating that the opinion of the nineteenth century was that the aether was on the whole stagnant and that, in the opinion of the author, "this doctrine has been subverted by the modern Principle of Relativity," seems to be of no value when put in such a magisterial form. The introduction (p. 29) of the equations of motion suffers from a defect shared with many text-books of putting a student in the puzzling position of being told by implication that these equations, on which the whole theory of dynamics is founded, are identities. I doubt the statement in the note on p. 34 that Lagrange's equations were first suggested in his early paper of 1760; in any case Prof. Whittaker should have given a rather more detailed justification for such a revolutionary statement.

The one other criticism which it seems to me should be made happens to be on that small part of the subject on which I have worked a good deal. It is not important from the point of view of the actual integration of mechanical equations, and it is chiefly in being an account of the latter subject that the great value of the present book consists. But the fact that no help is given to integration does not make a discussion unimportant in itself: witness Hamilton's partial differential equation. If, then, I speak about my own work, it is merely because I think that some points were brought forward in it which are not unimportant—and relevant even to Prof. Whittaker's own treatment. I think I can point out this in a few words. In the note on p. 249 there is a reference to some work by Voss in which an intention was announced of applying Hölder's process of thought to the case of general co-ordinates. In a paper in the *Mathematische Annalen* for 1908, which was the result of four years' work and controversy, I showed that Hölder's method is not used by Voss, and gave the approximate extension of Hölder's method. This extension does not happen to be immediately obvious and it brings out

the difference in the process of "variation" used in Hamilton's principle and the principle of least action and in each one when the dynamical system ceases to be holonomic, which is nowhere pointed out by Prof. Whittaker (pp. 245-50), although Hölder's extension of both these principles, which was carried out by him for rectangular co-ordinates alone, to the case of non-holonomic conditions is supposed to be described. However, all these criticisms do not touch the very great value of the book which has been and will be the chief path by which students in English speaking countries have been and will be introduced to modern work on the general and special problems of dynamics. PHILIP E. B. JOURDAIN.

Mathematical Monographs. Edited by M. MERRIMAN and R. S. WOODWARD. No. 17.

Lectures on Ten British Mathematicians of the Nineteenth Century. By ALEXANDER MACFARLANE. First Edition. First Thousand. Pp. 148. 5s. 6d. net. (Chapman and Hall.)

The present volume, we are delighted to hear, is the first instalment of twenty lectures given at Lehigh University some years before the author's lamented death. Macfarlane belongs to the little band of men who were proud to own allegiance to their teacher Tait, and he will be best known to a large number of mathematicians as a founder, as an industrious and active official, and ultimately as President of the *International Association for the Promoting the Study of Quaternions.*

It is much to be regretted that proper attention was not paid to the editing of the present volume. The publishers could have easily secured a competent person at least to identify easily verifiable quotations; more than one of them contain many misreadings, *e.g.* in Maxwell's line "around his head in *endless* cycles run," where Maxwell no doubt had his own reason for preferring "ceaseless," and a few lines further down where the sentence becomes sheer nonsense: "Above the host where your emblazoned rolls," where "wave" should take the place of "where." We are sure that Cayley was neither so discourteous nor so waggish as to allude in his own Presidential Address at a British Association meeting to the "providential" addresses of his predecessors (p. 69), and Tait must have had an impossible cross between the great giants of the city of London in his mind if ever he—a Scotchman and steeped in the Scriptures—was responsible for "He spared Agog, and the best of the sheep." Passages which are quotations are quoted as the words of the lecturer. A note on p. 82 to the well-known saying of Newton: "If Mr. Cotes had lived we might have known something," which apparently had for the moment escaped Macfarlane, should have been added. Letters are blind, and are omitted with irritating frequency (cf. pp. 87-88) or unnecessarily inserted, as in the generally abandoned form "Clairault," pp. 10, 35 and 147; "mathematicas," p. 65; words are omitted even where the sense is obviously destroyed (cf. "won't do" for "this won't do," p. 88). We have J for I in the anagram which, following ancient custom, was sent to *Nature* by Tait under the name of West, after to be explained in "The Unseen Universe." We have "plain" for "plane" (p. 87), "Lionville" for "Liouville," * within half a dozen lines of each other "Leibnitz" and "Leibniz"; an unintelligible notation is attributed to the German philosopher, and any decent compositor would be ashamed to leave uncorrected a reference to a "sovereing" and a feather. On p. 120, for 1872 read 1892. To call George Ticknor a scientist (p. 43) is a curious inversion of the current blunder which consistently confuses a "scholar" and a "savant" (what shall we call a savant who is also a scholar?). But poor Macfarlane was not responsible for most of these blemishes, and let us say at once that the volume is one which mathematicians who care to know something of the personalities of those who contributed so much to the progress of the science will read with pleasure almost unalloyed.

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* The *n* and *u* are fatal to unsuspicious compositors. A victim draws our attention to Ronth, Enler, Darbonx, Pinsenx, ...

The present instalment of these lectures is confined to studies of ten pure mathematicians: Peacock, De Morgan, William Rowan Hamilton, Boole, Cayley, Clifford, H. J. S. Smith, Sylvester, Kirkman and Todhunter. Of De Morgan we have the *Memoirs* published by his wife Sophia in 1882. We have the definitive life of Hamilton by Robert Perceval Graves. Frederic Pollock contributed a charming prefatory notice to the collection of Clifford's *Lectures and Essays* published in 1879; and H. J. S. Smith's appreciatory introduction to the *Scientific Papers* (1882) is well known. We might have expected a biography of Boole from the pen of his gifted wife, but a certain instability of temperament, which persisted after the guiding hand of her husband was withdrawn, unfortunately prevented this. For the rest, as far as we are aware, we must go in the main to short notices attached to editions of *Collected Works*, or to the obituaries in the *Proceedings* of such bodies as the Royal Society, the London Mathematical Society, or the Royal Astronomical Society. In the case of Kirkman, indeed, the dozen or so pages in this volume will be to a great extent a revelation, and we have met with many mathematicians who were barely aware of his existence. Of Todhunter we have three very characteristic papers in the *Cambridge Review*, vol. v. (1883-4), in which, however, we seem to see far more of the picturesque personality of the writer, dear old "Johnny Mayor," than of the subject of his discourse, the vivid touches it contains reminding us of Johnson's phrase on "reading Shakespeare by flashes of lightning." Of Todhunter, now that Besant is gone, there will be few who can say, as Professor Clifton writes to us, "I was introduced to Boole in his rooms." There are many still with us whose intimacy with some of the others would entitle them to write upon such attractive subjects. But we have said enough on the point to show that for the most part we are grateful for what the author has written upon these great figures in the mathematical history of early and mid-Victorian times.

It is interesting to note that Peacock's abilities were unsuspected in his early life. Such reputation as he had was due to his daring feats of climbing rather than to any promise in the class-room. Cayley was a mountaineer and Clifford an incomparable gymnast. In 1869 the latter writes from Cambridge: "I am at present in a very heaven of joy because my corkscrew was encored last night at the assault of arms; it consists in running at a fixed upright pole, which you seize with both hands and spin round and round, descending in a corkscrew fashion." A friend writes: "I am appalled now to think that he climbed up and sat on the cross bars of the weather-cock on a church tower, and when by way of doing something worse I went up and hung by my toes to the bars he did the same." The rest of the ten were not athletes, though Sylvester's use of the sword-cane was on one occasion dexterous enough to send him back from America to England without waiting to pack his books (p. 108). De Morgan was an exquisite flautist; Kirkman formed out of the roughest material "a parish choir of boys and girls who could sing at sight any four part song put before them." Todhunter knew two tunes, "one was 'God save the Queen,' the other wasn't. The former he recognised by the people standing up."

Among the men of his time at Cambridge, Peacock stood next in reputation to Whewell and Sedgwick. After a preliminary sketch of the conditions obtaining in the England of the opening of the nineteenth century, the lecturer traces the part played by Peacock in the work of reform, and indicates the manner in which, by "silent perseverance only," he and his colleagues in the new movement which they inaugurated hoped to "reduce the many-headed monster of prejudice and make the University answer her character as the loving mother of good learning and science." The quotation from the preface of Frend's *Principles of Algebra* (and here for *in*tractable read *un*tractable, for *soluble* read *solvable*, for *coulor* read *labour*) gives an idea of the gulf that was fixed between the British and the continental mathematicians. His published work is of considerable intrinsic and of great historical interest, and it is certainly worth while considering whether it will not be well to reprint his article on "Arithmetic" from the *Encyclopaedia Metropolitana*, which De Morgan graced with the epithet "most valuable," and to which he paid the "most undoubted compliment which can be paid to any work"

(*Arithmetical Books*, pp. xvii, 91). And there is also a masterly report to the meeting of the British Association of 1833 which should be rescued from an undeserved oblivion. Of De Morgan there is not much that is new, but all will endorse the recommendation of Tait to Macfarlane: "If you want to read something entertaining get De Morgan's *Budget of Paradoxes* out of the library." And the advice was taken to such effect that the lecturer dubs it De Morgan's "most" unique work!

Bishop Brinkley, the quondam astronomer, of whom Lord Norbury remarked that he had to thank his stars for his episcopal rank, said, so Maria Edgeworth tells us: "Young Mr. Hamilton may be a second Newton," which is a slight variant from Mr. Macfarlane's less comprehensive: "This young man, I do not say will be, but is, the first mathematician of his age." Sedgwick's estimate of the great Irishman was even higher—that Hamilton possessed within himself powers and talents perhaps never before combined within one philosophical character. The epitaph Hamilton desired for himself was ἀνὴρ φιλόπονος καὶ φιλαλήθης. As might be expected from a pupil of Tait, the importance attached to Hamilton's *Quaternions* by the author overshadows his work in other fields. It is interesting to remember that Cayley, while a pupil at the bar, went over to Dublin, and sat next to Salmon at Hamilton's first course of lectures on Quaternions. Salmon's appreciation of his companion twelve years before the death of Cayley (*Nature*, Sept. 20, 1883) is well worth reading for many reasons. We may select three "nuggets," one of which has escaped Mr. Macfarlane. The tripos list of 1842 was placed in Cayley's hands when on the top of the coach on a night journey to London, but he waited for the morning light to discover that he was the Senior Wrangler of that year.* He applied to Mr. Christie to be numbered among the pupils of that eminent conveyancer, modestly suppressing any mention of the fact of his antecedents, and only after some cross-examination did Christie find that he had before him a Senior Wrangler and a Fellow of Trinity. And last, "He is a good linguist, and, as was said of Mõltke, there are few European languages in which he does not know how to hold his tongue." Cayley "was invaluable as a mathematical referee." We hope to print some day in "Gleanings Far and Near' one of his reports, picked up by chance from its precarious enshrinement in a *Salmon's Conics* on a bookstall. Mr. G. B. Mathews refers (*Nature*, May 17, 1917) to Cayley's Presidential Address to the British Association of 1883, remarking that from the lecturer it receives "a proper amount of attention." We hope to reproduce in the *Gazette* from time to time many purple passages from addresses such as this, which are buried in reports and the like, unlikely to be disinterred but by those who have large libraries within reach.

The anagram referred to on p. 89 was signed "West," which stood for We, S(tewart) T(ait), and it is to be found in *Nature*, Oct. 15, 1874. The reply was composed by Clifford, except a page or two at the beginning, "at a single sitting which lasted from a quarter to ten in the evening till nine o'clock the following morning." Clifford's frankness and vigour of speech were naturally "offensive to many who held views as to religion in which he could not share," and reference might have been made to the somewhat cruel caricature which resulted from this in his later days when he figured as "Mr. Saunders" in W. H. Mallock's *New Republic*.

Of the ten pure mathematicians, Smith was, as Mr. G. B. Mathews says, the Admirable Crichton. In the obituary notice written by Spottiswoode (*Nature*, Feb. 22, 1883) the following passage is quoted from a letter of Huxley: "Henry Smith impressed me as one of the ablest men I ever met with; and the effect of his great powers was almost whimsically exaggerated by his extreme gentleness of manner, and the playful manner in which his epigrams were

* Another instance of real or assumed indifference of this kind is that of Dean Cowie, Senior Wrangler (1839) in a great Johnian year, the first four Wranglers, Cowie, Frost, Colson, Reyner, all being from St. John's. Cowie was one of the few Senior Wranglers the foundation of whose later distinction was laid by French hands. He drove up to the Senate House in a dog-cart, and enquired from his seat of the crowd: "Who's at the top?" To the answer: "Cowie!" he replied with the utmost nonchalance: "I thought he would be," and drove on.

scattered about. They were so bright and sharp that they transfixed their object without hurting him. I think that he would have been one of the greatest men of our time, if he had added to his own wonderfully keen intellect and strangely varied and extensive knowledge the power of caring very strongly about the attainment of any object," and Spottiswoode suggests that Smith's care for the attainment of an object was measured "rather by his estimate of its ultimate value than by its present advantage." His papers were remarkable for their exquisite finish, and Clifford's remark on reading a paper by Hesse has more than once been applied to the work of H. J. S. Smith: "this is reading poetry." The reader may add to the account of the 3000 francs prize (pp. 99, 105) that the motto he adopted for his paper was:

Quotque, quibusque modis possint in quinque resolvi
Quadratos numeri pagina nostra docet.

The story is also told in *Nature*, April 12, 1883, pp. 538 and 564. Curiously enough the three—Spottiswoode, Huxley and H. J. S. Smith—were simultaneous recipients of honorary degrees at Cambridge in 1879; the Public Orator's speeches are given in full in *Nature*, June 19, of that year.

The lines quoted in this lecture (p. 98) occur in Clerk Maxwell's address to Section A of the British Association, 1870, and it would be interesting to know their source:

"Where never creeps a cloud or moves a wind,
Nor ever falls the least white star of snow,
Nor ever lowest roll of thunder moans,
Nor sound of human sorrow mounts, to mar
Their sacred everlasting calm."

Huxley again appears, this time in the lecture on Sylvester, in connection with the moment when for once the great master of polemic met his match. In the pages on linkages, the Peaucellier mechanism for drawing a straight line is mentioned. Eight years after Sylvester's death, and three years after this lecture was delivered, attention was drawn by Mr. G. T. Bennett (*Phil. Trans*. 1905, vi. S. ix. p. 803) to the fact that Peaucellier had been anticipated by Sarrut, one of his own countrymen, who approached the problem by a different method (*Comptes Rendus*, vol. 36, 1853, p. 1036). Attention might have been called to the fertility of Sylvester's imagination as shown in his invention of appropriate terms in the course of his researches (did he not call himself a very Noah in this respect?). It has been suggested that the most felicitous of all was the use of "alternants" in the theory of determinants.

Macfarlane refers to the fact that Sylvester prided himself on his poetical powers, but does not allude to his Latin verse. Thanks to the kindness of Mr. A. O. Prickard, "Novi Collegii socius, vir amicissimus tam mitis ac benevolens quam lepidus et eruditus," and of Mr. P. E. B. Jourdain, we have before us a *Corolla Versuum*, for private circulation, with, needless to say, a revised edition with numerous additional notes, and dedicated: Cantatrici eximiae | nec minus habili ad delineandum manu | in omni genere artium verae Atheniensi | a Professore Saviliano Geometriae apud Oxonienses | et eodem Novi Collegii Socio | summisse oblata. Idibus Octob: 1895.

QUOD FELIX FAUSTUMQUE SIT.

We give one epigram:

URBI ET ORBI. Lond: 1865.

Quum theorema Newtoni de radicibus imaginariis equationum per longos annos non demonstratum, demonstrassem.

Ortae ex Cartesio, quam Newtonus insuper auxit,
Doctrinae en! demum fons et origo patent.

Which is Englished thus, thirty years later:

TO OXFORD AND THE WORLD.

Sprung from Descartes, which Newton to a higher level bore,
See now the fount and source lie open of that lore.

Or, again, Descartes' and Newton's law lay hid in night :
Heaven touched my heart with fire, and all was light.*

And a footnote : Ut lacus Nili ab ortus loco ita *fons* ab *origine* differt ; idcirco *patent* non *patet* iudicio meo legere fas est. Hoc distichon quum ex memoria excidisset (nam publicatum fuit in fasciculo inscripto (*Proceedings of London Mathematical Society*, anno 1865), revocavi admonitus ab Admirali Jonquières dum mense Augusto huius anni inter socios Instituti Franciae convivarer.

The writer of this notice possesses Kirkman's own copies of Plücker's *System der analytischen Geometrie* (1835), *System der Geometrie des Raumes* (1846) and the *Theorie der algebraischen Curven* (1839). These are bound in one volume, and from the inscriptions we see that he became possessed of the first in 1848, and of the second in the next year. He apparently had a habit of annotating his books, for the most part in a shorthand, possibly of his own. The volume contained two drafts of a relic not without its interest, a letter sent to a correspondent whose name, as the journalist says, does not transpire. The longer of the two is as follows :

Croft Warn[n], July 8, 1851.

My dear Sir,

As you have no books, I send you my slight improvement of Plücker's way of proving the number of double tangents.

Take any curve of n^{th} order $F(x, y)=0=U$ and diff[s],

$$F(x, y)=0=U \text{ and diff}^{s},$$
$$F'+F_1y'=0,$$
$$F_1y''+F''+2F_1'y'+F_{11}y'^2=0.$$

Then $$F_1^2F_{11}-2F'F_1F_1'+F_1^2F''=0=V,$$

a curve of $3n-4$ degree, is the locus of all the points of U for which $y''=0$ or the rad. cur. is infinite. If these points are at a finite distance they must be points of inflection on U. The number of these is just $n\,.\,\overline{3n-4}-\phi n$, $\frac{1}{2}\phi n$ being the number of asymptotes common to V and U; now $\phi n=4$ when $n=2$, since the points of inflection are nil. ϕn is $\therefore$ either $2n$ or $n+2$; but ϕn is always even, $\therefore$ it is $2n$; and the points of inflection are $n\,.\,\overline{3n-6}$ for $U=0$.

The number of tangents w you can draw to U from any point $w \parallel$ to any line is $n\,.\,\overline{n-1}$. These lines are determined by the equations

$$y'=a,$$
$$F=0=U,$$
$$F'+F_1y'=0=W,$$

and you draw the lines from your point to the $n\,.\,\overline{n-1}$ intersections of U and W; but this supposes that F, F', F_1 are not $=0$ together. If they are, the curve U has so many double points, and since V and dV equal 0 in this case, each of which makes two intersections of U and W, and reduces the possible proper tangents by 2. That is, the *class* of U is reduced by 2 for every double point on U. The order m of $R=0$, the polar reciprocal curve of U, is equal to the class of $U=0$ and $v\,.\,v$. Further, the number of tangents possible to draw is reduced by 3 for every cusp on U; for if the oval or loop vanishes, it becomes a cusp : now the line through the double point represents a pair of tangents practicable from p, and when the loop vanishes, it will represent 3 of them. Every point of inflection on U makes a cusp on R. The class of R is $m\,.\,\overline{m-1}$, or $n\,.\,\overline{n-1}(n\,.\,\overline{n-1}-1)$, but this is reduced by the $3n(n-2)$ cusps of R by $3\,.\,3n\,.\,\overline{n-2}$, or the class of R is then reduced to

$$\overline{n^2-n}\;\overline{n^2-n-1}-9n^2+18n \quad\text{or}\quad n^4-2n^3+n-9n^2+18n=n(n^3-2n^2-9n+19),$$

but this ought to be n; there is thus a further reduction of $n(n^3-2n^2-9n+18)$, which is due to the double point of R. (? These are)† number $\frac{n}{2}\,.\,\overline{n-2}\;\overline{n^2-9}$ and (? so)† the double tangents of U are of this exact number.

Among mathematicians we can recall but Leibniz and Kirkman who have paid serious attention to mnemonics. The latter was greatly struck by the

* The last is a memory from Pope's epitaph :—

"Nature and Nature's laws lay hid in night ;
God said, 'Let Newton be'; and all was light."

† Illegible.

merits of the system set forth by Richard Grey, the friend of Dr. Johnson, in his *Memoria Technica* (1730), of which various editions appeared up to 1861. Kirkman thought but little of Feinagle and Coglan, whose systems had a fleeting popularity, and declared such men to be worthy of chairs in the University of Laputa. "Their cumbrous inventions are about as fit to be compared, for elegance and speed, with the ἔπεα πτερόεντα of Richard Grey, as a Dutchman's ox-wagon in Kaffirland with a nobleman's chariot in Middlesex." So he published his *First Mnemonical Lessons in Geometry, Algebra, and Trigonometry* (John Weale, 1852). The key is self-explanatory :

b	d	t	f	l	s	p	k	n	z
1	2	3	4	5	6	7	8	9	0
a	e	i	o	u	au	oi	ei	ou	y

Thus 1851 is akla, or beila, or akub. The number π = tafaloudsutuknoint $= 3{\cdot}14159265358979\mathbf{3}$; $\sqrt{2} = 1{\cdot}4142135623731$, and is bobodatusdipta.

The example $\cot A \sin C + \cos b \cos C = \cot a \sin b$, which becomes

cot *Ang* si *Cang* and co *b* co *Cang* are cot *a* si *b*,

is not in the book mentioned above, and is probably in the paper of 1851 in the *Proceedings of the Literary and Philosophical Society of Manchester*. Kirkman's "Uncle Penyngton" is one of those clergymen "of mature knowledge, recognised ability, and blameless character," in whose case all suspicion on the part of the pupil is "irrational, and manifests a want of the power of appreciating evidence," as Todhunter would say. His two pupils, Richard and Jane, are chips from the old block. Jane's reply to an explanation of the mysterious $\sqrt{-1}$ by her revered preceptor is what one would expect : "So then whatever mystery or appearance of contradiction there may be here, it springs not from the answer of the oracle, but from the ignorance of the interrogator. His duty is not to cavil at the response, but to go away ashamed of himself and wondering." Uncle Pen. shared the prejudices of his age, for he says of mathematics, that it is a study in which, "My dear Jane . . . I should be sorry to see you deeply engaged." Richard shares little of the individuality of his charming sister. This is the book which De Morgan called "the most curious crotchet I ever saw." After all, one may over-estimate the soothing effect of jingles such as

RoóP slĕb .˙s̆ . sléc . slá,
Is Síne Bang hálf cá,
Is CHáSHca, is Area.

And who, by the following, would be kept from tripping in expanding $(1+r)^n$?

Sŭtón (ŭn *r*) ? wrīte frs̆ ŏn *r* ;
Theñ *r* to ī you multiply
By (i n-bácks bȳ i fags) :
If *n* hăs dĕn . é,
Put *r* vī (ré) ; top dits wĕd e.

After this one is not altogether astonished to see : "You will now find no difficulty. . . ."

It is interesting to note that the famous problem of the fifteen school girls, published in the *Lady's and Gentleman's Diary* for 1850 (Query VI. p. 48, and 1851, p. 48), was but an off-shoot from a paper Kirkman printed in the second volume of the *Cambridge and Dublin Mathematical Journal*. When the *Diary* went the way of all such things, the *Educational Times* stepped into the gap, and in the columns of the *Reprint* the name of Kirkman was for many years in evidence. We think he was the first to set his problems in (generally) amusing verse, and we seem to remember that in the long run Mr. Miller had to implore Kirkman's imitators to send their queries in plain prose.

Though sorely tempted we have not drawn upon the many good stories that are told in these lectures. A perusal of these pages will prove to the layman that mathematicians are not necessarily poor men of business, and that they are not as a rule devoid of humour.

One point in the lecture on Kirkman must not be passed over without echoing with Mr. G. B. Mathews his question to the responsible officials of the

Royal Society. What has become of the unpublished part of Kirkman's monumental paper on the "Complete Theory of the Polyhedra," of which but one-tenth has been printed? The subject is closely connected with group theory, and for a long time Kirkman was, if not the foremost, at least a very considerable figure among British contributors to the development of what was to prove "one of the most fundamental and fruitful notions in the whole range of our science."

It was given to Macfarlane to lay these garlands on the tombs of men, most of whom he had known, and all of whom he honoured. His hearers little thought that the volumes containing his lectures would in their turn serve so soon to remind us of one who has left behind him memories, fragrant and appreciative, on both sides of the great ocean which divided us from his manifold activities.

Differential and Integral Calculus. By C. E. Love. Pp. xviii + 343. 9s. net. 1916. (The Macmillan Company.)

English teachers may derive some advantage from the consideration of this recent importation from an American University. The author has had more particularly in view the requirements of the student of engineering, and "the non-geometric applications are taken systematically from one subject, mechanics." That, however, is not uncommon in modern text-books, and the more notable points in connection with this book to which the attention of the teacher may be drawn are the stress that is laid on the checking of the work, and the inclusion of a few pages on the line integral. About one-sixth of the book is given to differential equations of the first and of higher order, with applications to mechanics. Dr. Love has found, it would seem, that the class of student with which he has had to deal consists of pupils in varying stages of development. It is not likely that to those who are told that one way round an obstacle is to believe that "it can be shown that . . ." are in the same stage as those who are assured that "it is evident that . . ." Such difficulties, however, are perhaps inherent in the situation. "In spite of obvious difficulties, a chapter embodying a first treatment of centroids and moment of inertia is introduced before multiple integrals have been defined. By this arrangement the student is brought to realise the fact that in most cases of practical importance mass-moments of the first and second orders can be found by simple integration, whereas from the usual treatment he gets exactly the opposite idea." The treatment is generally clear and straightforward, but the private student will find that to a large number of the questions no answers are given—probably a wise practice where the teacher's aid is available.

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